Quantum Channels: Different Representations

Recap: Last week focussed on the Kraus representation

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$$

$$\text{s.t. } \sum_i A_i^\dagger A_i = I$$

This week will cover two other representations

- Stinespring Dilation
- The choi representation

Stinespring Dilation

Interacting a system and an environment & then ignoring the environment (tracing it out) induces a non-unitary channel on the system $\rightarrow$ (we saw this in mathematica problem sheet)

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eg. Suppose we start with $\rho_S \otimes |0\rangle\langle 0|_E$

& Evolve under $U = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$

$$U \rho_s \otimes |0\rangle\langle 0|_E U^\dagger = \rho_s^{00} |0\rangle\langle 0| \otimes U_0 |0\rangle\langle 0| U_0^\dagger + \rho_s^{11} |1\rangle\langle 1| \otimes U_1 |0\rangle\langle 0| U_1^\dagger + \rho_s^{01} |0\rangle\langle 1| \otimes U_0 |0\rangle\langle 0| U_1^\dagger + \rho_s^{11} |1\rangle\langle 1| \otimes U_1 |0\rangle\langle 0| U_0^\dagger$$

Read as system has a conditional action on the environment

$$\rho_s = \mathrm{Tr}_E\left(U(\rho_s \otimes |0\rangle\langle 0|_E) U^\dagger\right)$$

$$= \rho_s^{00}|0\rangle\langle 0| + \rho_s^{11}|1\rangle\langle 1| + c\,\rho_s^{01}|0\rangle\langle 1| + c^*\rho_s^{10}|1\rangle\langle 0|$$

$c := \langle 0|U_1^\dagger U_0|0\rangle \equiv \langle \psi_1|\psi_0\rangle$

assuming for simplicity c is real

$$\equiv p\,\rho + (1-p)\,Z\rho Z$$

$$= p\,\rho +$$

$$(1-p)\ \rho_s^{00}|0\rangle\langle 0| + \rho_s^{11}|1\rangle\langle 1| - \rho_s^{01}|0\rangle\langle 1| - {}^*\rho_s^{10}|1\rangle\langle 0|$$

$$= \rho_s^{00}|0\rangle\langle 0| + \rho_s^{11}|1\rangle\langle 1| + \underbrace{(2p-1)}_{c}\left(\rho_s^{01}|0\rangle\langle 1| + \rho_s^{10}|1\rangle\langle 0|\right)$$

This is known as the "(non-completely) Dephasing channel"
It kills off coherence turning a quantum state into a classical mixture.

If $c = 0 \iff p = 1/2$ we end up with a perfect classical mixture → i.e. the completely dephasing channel we saw last time...

"System acting with measurement / environment such that environment is mapped to orthogonal output states leads to loss of coherence"
~ Decoherence

But the converse is also true...

- Similarly to how tracing out part of an entangled state lead to a mixed state

  & any mixed state can be purified into a part of a larger entangled system

→ Any quantum channel can be written in terms of a unitary acting on an enlarged system.

Church of the Larger Hilbert space (again)

Stinespring's Dilation Theorem

Any quantum channel can be written in the form

$$\mathcal{E}(\rho) = \mathrm{Tr}_E\left(U \left(\rho \otimes |0\rangle\langle 0| \right) U^\dagger \right)$$

Unitary

Proof*: Let $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$ s.t. $\sum_i A_i^\dagger A_i = I$

We need: $U |0\rangle |\psi\rangle = \sum_i |i\rangle A_i |\psi\rangle \quad \forall \psi$

if that's the case:

$$\mathrm{Tr}_E\left(U (|0\rangle\langle 0| \otimes |\psi\rangle\langle\psi|) U^\dagger \right)$$

$$= \sum_{ij} \mathrm{Tr}_E\left(I \otimes A_i (|i\rangle\langle j| \otimes |\psi\rangle\langle\psi|) I \otimes A_j^\dagger \right)$$

$$= \sum_i A_i \rho A_i^\dagger = \mathcal{E}(\rho)$$

$$|0\rangle|\psi\rangle = \begin{pmatrix} 1 \\ \\ \end{pmatrix} \otimes |\psi\rangle = \begin{pmatrix} |\psi\rangle \\ 0 \\ \vdots \end{pmatrix} \qquad |1\rangle|\psi\rangle = \begin{pmatrix} 0 \\ |\psi\rangle \\ 0 \\ \vdots \end{pmatrix}$$

$$U|0\rangle|\psi\rangle = \begin{pmatrix} A_0 & \boxed{\begin{matrix}\text{free to}\\ \text{pick}\\ \text{to}\\ \text{make}\\ \text{unitary}\end{matrix}} \\ A_1 & \\ A_2 & \\ \vdots & \end{pmatrix} \begin{pmatrix} |\psi\rangle \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} A_0|\psi\rangle \\ A_1|\psi\rangle \\ A_2|\psi\rangle \\ \vdots \end{pmatrix} = \sum_i |i\rangle A_i|\psi\rangle$$

The Choi Representation

First introduce the notion of vectorisation ↗ simple but powerful trick

Say $A = \sum_{ij} a_{ij} |i\rangle\langle j|$

$$|Vec(A)\rangle = \sum_{ij} a_{ij} |ij\rangle$$

eg. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow |00\rangle + |11\rangle$

$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \rightarrow a|01\rangle + b|10\rangle$

One of the reasons vectorization is useful is because of the following identity:

Central Vectorization Identity

$$\star \quad |Vec(AXB)\rangle = A \otimes B^T |Vec(X)\rangle$$

Proof: $$A \otimes B^T |Vec(X)\rangle = \sum_{iji'j'} A_{ij} |i\rangle\langle j| \overbrace{\left(B_{i'j'} |i'\rangle\langle j'|\right)^T}^{B_{i'j'}|j'\rangle\langle i'|} x_{kk'} |kk'\rangle$$

$$= \sum_{iji'j'} A_{ij} B_{i'j'} x_{kk'} |ij'\rangle \delta_{jk} \delta_{i'k'}$$

$$= \sum_{ii} \underbrace{A_{ik} X_{ki'} B_{kj'}}_{C_{ij'}} |ij'\rangle$$

$$= \text{Vec}(C) \quad \text{with } C = AXB$$

Ok - so why is this useful?

One reason it's useful is we can use it to show that:

Super useful identity:

$$(U^\dagger \otimes I)|\phi^+\rangle = I \otimes U^* |\phi^+\rangle$$

Proof: $|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle = \frac{1}{\sqrt{d}} |\text{Vec}(I)\rangle$

$$(U^\dagger \otimes I)|\phi^+\rangle = \frac{1}{\sqrt{d}} (\underset{A}{U} \otimes \underset{B^T}{I}) |\text{Vec}(I)\rangle$$

$$= \frac{1}{\sqrt{d}} |\text{Vec}(AB)\rangle$$

$$= \frac{1}{\sqrt{d}} |\text{Vec}(U^\dagger)\rangle$$

$$= \frac{1}{\sqrt{d}} |\text{Vec}(\underset{I}{A} \underset{U^\dagger}{B})\rangle$$

$$= (I \otimes U^*)|\phi^+\rangle$$

"Alice apply U to her half a Bell state is equivalent to Bob applying U^* to his...."

This trick can be used to reduce circuit depths.

e.g. $(UV^\dagger \otimes I)\,|\phi^+\rangle$

$= (U \otimes I)(V^\dagger \otimes I)\,|\phi^+\rangle$

$= U \otimes V^* \,|\phi^+\rangle$ ← assuming U is depth D & V is depth D have reduced depth by ½!

Choi-Jamiolkowski Representation

For any quantum channel $\mathcal{E}$ we define the Choi state associated to the channel as

$$J(\mathcal{E}) := \mathcal{E} \otimes I\left(\underbrace{|Vec(I)\rangle}\langle Vec(I)|\right)$$

you see definitions both with/without the factor of dimension included.

$= |Vec(I \otimes I)\rangle$
$= \sum_i |ii\rangle$
$\propto |\phi^+\rangle$

The Choi state uniquely specifies the quantum channel. If $\mathcal{E}$ has Kraus representation $\{A_i\}$ then the Choi state is given by

$$J(\mathcal{E}) = \sum_i |Vec(A_i)\rangle\langle Vec(A_i)|$$

To see this final claim: $J(\mathcal{E}) = \sum_i (A_i \otimes I)\,|Vec(I)\rangle\langle Vec(I)|\,(A_i^\dagger \otimes I)$

via ☆ $= \sum_i |Vec(A_i)\rangle\langle Vec(A_i)|$

Examples

$\mathcal{E}(\rho)$ = completely dephasing channel (on single qubit) $\quad A_0 = |0\rangle\langle 0|$, $A_1 = |1\rangle\langle 1|$

$$|\mathrm{Vec}(I)\rangle = |00\rangle + |11\rangle$$

$$J(\mathcal{E}) = (\mathcal{E}\otimes I)\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|\right)$$

$$= \mathcal{E}(|0\rangle\langle 0|)\,|0\rangle\langle 0| + \mathcal{E}(|1\rangle\langle 1|)\,|1\rangle\langle 1| + \underbrace{\mathcal{E}(|0\rangle\langle 1|)}_{0}\,|0\rangle\langle 1| + \overbrace{\mathcal{E}(|1\rangle\langle 0|)}^{0}\,|1\rangle\langle 0|$$

$$= \underbrace{|00\rangle\langle 00|}_{|\mathrm{Vec}(|0\rangle\langle 0|)} + \underbrace{|11\rangle\langle 11|}_{|\mathrm{Vec}(|1\rangle\langle 1|)} \equiv$$

ie. this holds

$$\star \quad = |\mathrm{Vec}(A_0)\rangle\langle \mathrm{Vec}(A_0)| + |\mathrm{Vec}(A_1)\rangle\langle \mathrm{Vec}(A_1)|$$

What's the point in the Choi Representation?

1. Often ends up being easier mathematically / computationally to work in terms of states rather than channels.
   (e.g. in algorithms for learning / simulating channels)

2. It provides a way of finding the Kraus operators for a channel

Example $\quad \mathcal{E}(X) = \frac{1}{3}\left(\mathrm{Tr}(X)\,I + X^T\right)$ $\Big\}$ What are its Kraus operators?

$$J(\mathcal{E}) = (\mathcal{E}\otimes I)\left(\sum_{ij} |ii\rangle\langle jj|\right)$$

$$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

$$= \sum_{ij} \frac{1}{3}\left(\mathrm{Tr}(\cancel{|i\rangle\langle j|})^{\delta_{ij}} I + |i\rangle\langle j|^T\right) \otimes |i\rangle\langle j|$$

$$= \frac{1}{3}\left(\sum_i I \otimes |i\rangle\langle i| + \sum_{ij} |ji\rangle\langle ij|\right)$$

$$= \frac{1}{3}(I + \mathrm{SWAP})$$

$$|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|$$

$$\quad '' \qquad\qquad '' \qquad\qquad '' \quad + \quad ''$$

Note, if you didn't spot that SWAP could be decomposed like this, you could just find the eigendecomp. of $I + \mathrm{SWAP}$

$$= \frac{2}{3}\left(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|\right)$$

$$= \frac{2}{3}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |\psi^+\rangle\langle\psi^+|\right)$$

$\Rightarrow$ 2 different possible choices in kraus operators

<u>Option 1</u>:

$$|\mathrm{Vec}(A_0)\rangle = \sqrt{\tfrac{2}{3}}|\phi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle) \Rightarrow A_0 = \frac{1}{\sqrt{3}} I$$

$$|\mathrm{Vec}(A_1)\rangle = \sqrt{\tfrac{2}{3}}|\phi^-\rangle \Rightarrow A_1 = \frac{1}{\sqrt{3}} Z$$

$$|\mathrm{Vec}(A_2)\rangle = \sqrt{\tfrac{2}{3}}|\psi^+\rangle \Rightarrow A_2 = \frac{1}{\sqrt{3}} X$$

note $\sum_i A_i^\dagger A_i = \sum_i \frac{1}{3} I = I$ ✓

<u>Option 2</u>:

$$|\mathrm{Vec}(A_0)\rangle = \sqrt{\tfrac{2}{3}}|00\rangle \qquad A_0 = \sqrt{\tfrac{2}{3}}\,|0\rangle\langle 0|$$

$$|\mathrm{Vec}(A_1)\rangle = \sqrt{\tfrac{2}{3}}|11\rangle \qquad A_1 = \sqrt{\tfrac{2}{3}}\,|1\rangle\langle 1|$$

$$|\mathrm{Vec}(A_2)\rangle = \sqrt{\tfrac{2}{3}}|\psi^+\rangle \qquad A_2 = \frac{1}{\sqrt{3}} X$$

$$\sum_i A_i^\dagger A_i = \frac{2}{3} I + \frac{1}{3} I = I$$

General Recipe for finding Kraus Operators

1) Take $\mathcal{E}$ and compute $J(\mathcal{E})$
2) Find eigendecomposition of $J(\mathcal{E})$

$$J(\mathcal{E}) = \sum_k \lambda_k |e_k\rangle\langle e_k|$$

3) Find the matrix A_k such that

$$|vec(A_k)\rangle = \sqrt{\lambda_k}\, |e_k\rangle$$

4) The matrices $\{A_k\}$ are the smallest set of Kraus operators for $\mathcal{E}$.

Unitary mixing freedom in Kraus representation

We saw above (& previously) that multiple sets of Kraus operators can be used to describe the same channel. This can be understood within the Choi representation.

We have $J(\mathcal{E}) = \sum_i |vec(A_i)\rangle\langle vec(A_i)|$

but remember this is just a state & we can write it in its eigendecomposition or just as an ensemble decomposition.

i.e. $\mathcal{J}(\mathcal{E}) = \sum_i |Vec(A_i)\rangle\langle Vec(A_i)|$

$= \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$

$= \sum_i p_i |\phi_i\rangle\langle\phi_i|$

These are related by $\sqrt{p_i}|\phi_i\rangle = \sum_k u_{ik}\sqrt{\lambda_k}|\lambda_k\rangle$

see lecture 2

elements of a unitary/isometry

$|Vec(A_i)\rangle = \sqrt{\lambda_i}|\lambda_i\rangle$

$|Vec(B_i)\rangle = \sqrt{p_i}|\phi_i\rangle = \sum_k u_{ik}\sqrt{\lambda_k}|\lambda_k\rangle$

$= \sum_k u_{ik}|Vec(A_k)\rangle$

$\sum_{jj'}[B_i]_{jj'}|jj'\rangle = \sum_{k, jj'} u_{ik}[A_k]_{jj'}|jj'\rangle$

$\therefore [B_i]_{jj'} = \left[\sum_k u_{ik}A_k\right]_{jj'}$

$\equiv \boxed{B_i = u_{ik}A_k}$

Different possible sets of Kraus operators are related by unitary/isometric mixing